REFERENCES

[1] H. B. Dwight, "Tables of Integrals," The Macmillan Company, New York, N. Y.; 1934.
[2] A. Betz, "Konforme Abbildung," Springer-Verlag, Berlin, Germany; 1948. (Parallel strips, pp. 241–243; derivation in terms of elliptic functions.)
[3] F. Oberhettinger and W. Magnus, "Anwendung der Elliptischen Funktionen in Physik und Technik," Springer-Verlag, Berlin, Germany; 1949. (Parallel strips, pp. 63–64; exact solution in elliptic functions, graph for low-resistance range.)
[4] E. Weber, "Electromagnetic Fields—Theory and Applications—Mapping of Fields," John Wiley and Sons, Inc., New York, N. Y., Vol. I; 1950. (Parallel plates, pp. 333–338; 356–357; other references.)
[5] H. A. Wheeler, "The Capacitance of Two Parallel Wires of Different Diameter," Wheeler Labs., Inc., Great Neck, N. Y., Monograph No. 14; 1950. (Unusual formulas for balanced pair.)
[6] H. Kober, "Dictionary of Conformal Representations," Dover Publications, Inc., New York, N. Y.; 1952. (Parallel strips, pp. 190, 194–195; exact solution in elliptic functions.)
[7] F. Assadourian and E. Rimai, "Simplified theory of microstrip transmission systems," PROC. IRE, vol. 40, pp. 1651–1657; December, 1952. (Approximate formula for wide strip above a plane; error about 20 per cent for shape corresponding to parallel strips of equal width and separation.)
[8] H. A. Wheeler, "The Transmission-Line Properties of a Round Wire Between Parallel Planes," Wheeler Labs., Inc., Great Neck, N. Y., Monograph No. 19; 1954. Also. IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-3, pp. 203–207; October, 1955. (Formulas for synthesis in terms of exponential series.)
[9] ——, "Handbook—Lines and Guides," Wheeler Labs., Great Neck, N. Y., Rept. No. 522, pp. 9–10; December, 1954. (Similar formulas for parallel wide strips.)
[10] K. G. Black and T. J. Higgins, "Rigorous determination of the parameters of microstrip transmission lines," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-3, pp. 93–113; March, 1955. (Many references.)
[11] H. Meinke and F. W. Grundlach, "Taschenbuch der Hochfrequenztechnik," Springer-Verlag, Berlin, Germany; 1956. (Strip lines, p. 174.)
[12] S. Cohn, private communications; May–June, 1958. (Derivation equivalent to formula (81) in this paper.)
[13] W. von Koppenfels and F. Stallman, "Praxis der Konformen Abbildung," Springer-Verlag, Berlin, Germany; 1959. (Parallel strips in general, pp. 340–344; elliptic functions.)
[14] H. P. Kunzi, "Quasikonforme Abbildungen," Springer-Verlag, Berlin, Germany; 1960.
[15] K. S. Packard and R. V. Lowman, "Transmission lines and waveguides," in "Antenna Engineering Handbook," McGraw-Hill Book Company, Inc., New York, N. Y., ch. 30; 1961. (Strip lines, Fig. 30-7; references.)
[16] A. F. Harvey, "Microwave Engineering," Academic Press, London England; 1963. (Strip lines, pp. 407–415; references.)

Realizable Limits of Error for Dissipationless Attenuators in Mismatched Systems

D. C. YOULA, SENIOR MEMBER, IEEE, AND P. M. PATERNO, MEMBER, IEEE

Summary—**A tutorial exposition for the exact physical error limits due to mismatch for dissipationless attenuators is presented. The results given yield smaller errors than previous existing formulas due to the inclusion of the physical realizability constraint of passivity. Graphs are included for rapidly determining the largest error for a prescribed set of conditions. This work is based on an analysis prepared by D. C. Youla which had a limited circulation.**[1]

INTRODUCTION

FORMULAS for the errors resulting from mismatched generator and detector sections in the measurement of the attenuation of a single attenuator are well known.[2,3] However, with a single exception[4] none of these formulas takes into account the phase restrictions on the various coefficients due to the physical realizability constraint of passivity. The usual formulas exhibit limits obtained by choosing the worst possible phase combinations and, therefore, lead to unnecessarily large errors. In this paper a complete solution is presented for the *physical* error limits due to mismatch for *dissipationless* attenuators calibrated under the standard condition of *conjugate termination*. This includes, of course, the class of equal-resistance attenuators but is more general.

GENERAL BACKGROUND

In a previous publication,[5] a complete scattering description for a linear time-invariant $2N$ terminal net-

Manuscript received June 17, 1963; revised January 20, 1964. The work performed by D. C. Youla was sponsored by the Air Force Cambridge Research Laboratory, Office of Aerospace Research, Cambridge, Mass., under Contract No. AF-19(604)-4143.

D. C. Youla is with the Polytechnic Institute of Brooklyn, Brooklyn, N. Y.
P. M. Paterno is with PRD Electronics, Inc., Brooklyn, N. Y.

[1] D. C. Youla, "Limits of Errors in Mismatched Attenuators—Part I," Polytechnic Institute of Brooklyn, Brooklyn, N. Y., Memo. No. 68, PIBMRI-1042-62.

[2] C. G. Montgomery, "Technique of Microwave Measurements," McGraw-Hill Book Co., Inc., New York, N. Y., p. 824; 1947.

[3] R. W. Beatty, "Mismatch Errors in the Measurement of Ultrahigh Frequency and Microwave Variable Attenuators," NBS Res. Paper 2465, vol. 52, no. 1; January, 1954.

[4] L. O. Sweet and M. Sucher, "The available power of a matched generator from the measured load power in the presence of small dissipation and mismatch of the connecting network," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-5, pp. 167–168; April, 1957.

[5] D. C. Youla, "On scattering matrices normalized to complex port numbers," PROC. IRE, vol. 49, p. 1221; July, 1961.

work terminated with complex impedances z_N at each port was presented. The approach used was to define the normalized incident and reflected wave amplitudes a_K and b_K as linear combinations of the respective port voltages as shown below.

$$2\sqrt{r_K}\, a_K = V_K + z_K I_K \tag{1}$$

$$2\sqrt{r_K}\, b_K = V_K - \bar{z}_K I_K \tag{2}$$

where

$$r_K = \text{Real } z_K(j\omega) > 0.$$

V_K, I_K = voltage and current at port k, respectively, and a bar over a quantity denotes complex conjugate.[6] The relationship between a_K and b_K is defined by means of the linear matrix equation

$$b = Sa \tag{3}$$

where S the scattering matrix of N is normalized with respect to the n impedances

$$Z_1, Z_2, \cdots, Z_n.$$

If all ports of N except the kth port are closed on their respective normalization impedances and port k is driven by a generator with internal impedance z_K it can be shown that

$$\frac{b_K}{a_K} = S_{KK} = \frac{Z_K - \bar{z}_K}{Z_K + z_K} \cdot \tag{4}$$

In the above, S_{KK}, Z_K represent the input reflection coefficient and input impedance at port k under matched conditions. An immediate observation is that (4) admits a correct solution for the special case of conjugate termination, an answer which cannot be obtained from the usual formula obtained from standard normalization procedures.

In light of the above remarks, consider a situation in which a generator E with internal impedance z_{01} is connected directly to a load impedance z_{02} (Fig. 1). Let r_{01} and r_{02} denote the real parts of z_{01} and z_{02}, respectively. From first principles, P_{lo}, the average ac power delivered to the load is given by

$$P_{lo} = P_{mo} \cdot \left(1 - \left| \frac{z_{02} - \bar{z}_{01}}{z_{02} + z_{01}} \right|^2\right) \tag{5}$$

where

$$P_{mo} = \frac{|E|^2}{4r_{01}} \tag{6}$$

is the maximum available generator power, and bar denotes complex conjugate.

Suppose now that a passive 2-port N is interposed between z_{01} and z_{02} (Fig. 2). Denote the scattering matrix of N *normalized* to z_{01} at port No. 1 and z_{02} at port No. 2 by[3]

[6] The notation used in this paper is consistent with that used by Youla.[5]

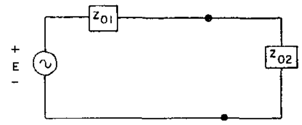


Fig. 1—A direct interconnection of generator and load.

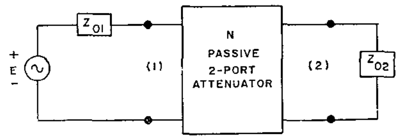


Fig. 2—Attenuator schematic.

$$S_o = \begin{bmatrix} s_{11}^o & s_{12}^o \\ s_{21}^o & s_{22}^o \end{bmatrix}. \tag{7}$$

The average power, W_{lo}, that is now delivered to z_{02} is

$$W_{lo} = |s_{21}^o|^2 P_{mo}. \tag{8}$$

The *attenuation* loss in decibels, A_o, attributable to the interposition of N, is by definition

$$A_o = 10 \log \frac{P_{lo}}{W_{lo}} \tag{9}$$

$$= 10 \log \left(1 - \left| \frac{z_{02} - \bar{z}_{01}}{z_{02} + z_{01}} \right|^2\right) + 20 \log \frac{1}{|s_{21}^o|} \cdot \tag{10}$$

Under conjugate terminations $z_{01} = \bar{z}_{02} = z_0$, say

$$A_o = 20 \log \frac{1}{|s_{21}^o|} \cdot \tag{11}$$

In the laboratory, the microwave engineer usually finds himself in the position of operating an attenuator, calibrated under a standard set of conditions, $z_{01} = \bar{z}_{02} = z_0$, between mismatched terminations z_g and z_l. The calibration procedure used here includes the case of matched resistive terminations but is more general.

Moreover, very little or no phase information is available concerning z_g and z_l. Usually the only data which is known to any degree of accuracy is embodied in the *magnitudes* of the mismatch reflection coefficients

$$\Gamma_g = \frac{z_g - z_o}{z_g + \bar{z}_o} \tag{12}$$

and

$$\Gamma_l = \frac{z_l - \bar{z}_o}{z_l + z_o} \cdot \tag{13}$$

If A represents the attenuation actually achieved under the conditions described above, the mismatch error is

$$\epsilon = A - A_o \text{ db.} \tag{14}$$

Since the attenuator setting is measured by $|s_{21}^o|$ ($A_o = 20 \log (1/|s_{21}^o|)$) the problem reduces to finding the largest value of $|\epsilon|$, for a prescribed set of values $|\Gamma_g|$, $|\Gamma_l|$ and $|s_{21}^o|$, subject to the limitation that N, z_g, z_l and z_0 are *passive*. In what follows, log a and ln a denote the logarithms of a taken to the base 10 and e, respectively.

Analysis

The first step is to derive an expression for ϵ. Let

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \tag{15}$$

denote the scattering matrix of N (the *same* attenuator) normalized to z_g at port 1 and z_l at port 2. Then, proceeding as before [see (10)],

$$A = 10 \log \left(1 - \left| \frac{z_l - \bar{z}_g}{z_l + z_g} \right|^2 \right) + 20 \log \frac{1}{|s_{21}|} \cdot \tag{16}$$

Now

$$\begin{aligned} 1 - \left| \frac{z_l - \bar{z}_g}{z_l + z_g} \right|^2 &= 1 - \frac{z_l - \bar{z}_g}{z_l + z_g} \cdot \frac{\bar{z}_l - z_g}{\bar{z}_l + \bar{z}_g} \\ &= \frac{(z_g + \bar{z}_g)(z_l + \bar{z}_l)}{(z_l + z_g)(\bar{z}_l + \bar{z}_g)} \\ &= \frac{(1 - |\Gamma_g|^2)(1 - |\Gamma_l|^2)}{|1 - \Gamma_g \Gamma_l|^2} \cdot \end{aligned} \tag{17}$$

The details appear in Appendix I. Before (16) can be used, a relationship between the elements of S and S_o is required. This is necessary since the calibration of the attenuator is described by the elements of S_o when normalized to z_{01} and z_{02} and is used between terminations z_g and z_l which in general are different from z_{01}, z_{02}. In Appendix II it is shown that

$$s_{11} = \frac{1 - \Gamma_g}{1 - \bar{\Gamma}_g} \cdot \frac{s_{11}^o + \bar{\Gamma}_g \Gamma_l s_{22}^o - \Gamma_l \Delta_o - \bar{\Gamma}_g}{D_o} \tag{18}$$

$$s_{12} = \frac{1 - \Gamma_l}{1 - \bar{\Gamma}_g} \cdot \left| \frac{1 - \Gamma_g}{1 - \Gamma_l} \right| \cdot \frac{s_{12}^o \sqrt{(1 - |\Gamma_g|^2)(1 - |\Gamma_l|^2)}}{D_o} \tag{19}$$

$$s_{21} = \frac{1 - \Gamma_g}{1 - \bar{\Gamma}_l} \cdot \left| \frac{1 - \Gamma_l}{1 - \Gamma_g} \right| \cdot \frac{s_{21}^o \sqrt{(1 - |\Gamma_g|^2)(1 - |\Gamma_l|^2)}}{D_o} \tag{20}$$

and

$$s_{22} = \frac{1 - \Gamma_l}{1 - \bar{\Gamma}_l} \cdot \frac{s_{22}^o + \Gamma_g \bar{\Gamma}_l s_{11}^o - \Gamma_g \Delta_o - \bar{\Gamma}_l}{D_o} \tag{21}$$

where

$$\Delta_o = \det S_o = s_{11}^o s_{22}^o - s_{12}^o s_{21}^o \tag{22}$$

and

$$D_o = 1 - \Gamma_g s_{11}^o - \Gamma_l s_{22}^o + \Gamma_g \Gamma_l \Delta_o. \tag{23}$$

From (20)

$$\left| \frac{s_{21}^o}{s_{21}} \right|^2 = \frac{|D_o|^2}{(1 - |\Gamma_g|^2)(1 - |\Gamma_l|^2)} \cdot \tag{24}$$

Combining (13) and (15) results in

$$\epsilon = 10 \log \left| \frac{s_{21}^o}{s_{21}} \right|^2 + 10 \log \left(1 - \left| \frac{z_l - \bar{z}_g}{z_l + z_g} \right|^2 \right) \tag{25}$$

which reduces to the following compact expression when (17) and (24) are substituted into (25):

$$\epsilon = 20 \log \left| \frac{D_o}{1 - \Gamma_g \Gamma_l} \right| \text{db.} \tag{26}$$

To find the largest value of $|\epsilon|$, $|\epsilon|_{max}$, it is necessary to determine the two extreme limits

$$\lambda_M = \left| \frac{D_o}{1 - \Gamma_g \Gamma_l} \right|_{max} \tag{27}$$

and

$$\lambda_m = \left| \frac{D_o}{1 - \Gamma_g \Gamma_l} \right|_{min} \tag{28}$$

subject to the physical realizability restrictions discussed above. If

$$\lambda_m \lambda_M \leq 1, \tag{29}$$

then

$$|\epsilon|_{max} = -20 \log \lambda_m = 20 \log \lambda_m^{-1}, \quad (30$$

but if

$$\lambda_m \lambda_M > 1, \tag{31}$$

then

$$|\epsilon|_{max} = 20 \log \lambda_M. \tag{32}$$

The choice of formula will depend, in general, not only on the *attenuator setting*, A_o, but also on the parameters $|s_{11}^o|$, $|s_{12}^o|$, $|s_{22}^o|$, $|\Gamma_g|$ and $|\Gamma_l|$. We shall see later that for lossless N (filters, cutoff attenuators, etc.) the three quantities A_o, $|\Gamma_g|$ and $|\Gamma_l|$ suffice.

Constraints

The passivity requirement on N is equivalent to stating that the three principal minors of the 2×2 hermitian matrix

$$Q_o = 1_2 - S_o^* S_o$$

are non-negative,[7] *i.e.*,

$$|s_{11}^o|^2 + |s_{12}^o|^2 \leq 1 \tag{33}$$

$$|s_{22}^o|^2 + |s_{21}^o|^2 \leq 1 \tag{34}$$

[7] H. J. Carlin, "The scattering matrix in network theory," IRE Trans. on Circuit Theory, vol. CT-3, pp. 88–96; June, 1956.

and

$$(1 - |s_{12}^o|^2)(1 - |s_{21}^o|^2) + (1 - |s_{11}^o|^2)(1 - |s_{22}^o|^2) \geq 1 \pm 2 \operatorname{Re}(s_{11}^o s_{22}^o \overline{s_{12}^o}\, \overline{s_{21}^o}). \quad (35)$$

If N is *lossless*, $1_2 - S_o^* S_o = O_2$ and the inequalities in (33)–(35) become equalities:

$$|s_{11}^o|^2 + |s_{12}^o|^2 = 1, \quad (36)$$

$$|s_{22}^o|^2 + |s_{21}^o|^2 = 1 \quad (37)$$

and

$$s_{21}^o \overline{s_{11}^o} + s_{22}^o \overline{s_{12}^o} = 0. \quad (38)$$

Eqs. (36)–(38) imply the equality of the magnitudes of front and back-end reflection coefficients:

$$|s_{11}^o| = |s_{22}^o|. \quad (39)$$

As a rule attenuators are lossy but there are situations in which lossless transforming networks play a role, and they will be discussed later on.

The most important realizability constraint is the inequality expressed in (35). To cast it in a more manageable form let

$$s_{11}^o = |s_{11}^o| e^{j\theta_{11}}$$

$$s_{12}^o = |s_{12}^o| e^{j\theta_{12}}$$

$$s_{21}^o = |s_{21}^o| e^{j\theta_{21}}$$

$$s_{22}^o = |s_{22}^o| e^{j\theta_{22}}. \quad (40)$$

Substituting in (35) and simplifying yields

$$(1 - |s_{12}^o|^2)(1 - |s_{21}^o|^2) + (1 - |s_{11}^o|^2)(1 - |s_{22}^o|^2) \geq 1 + 2|s_{11}^o s_{22}^o s_{12}^o s_{21}^o| \cos\beta \quad (41)$$

where

$$\beta = \theta_{11} + \theta_{22} - \theta_{12} - \theta_{21}. \quad (42)$$

For *reciprocal* attenuators, $s_{12}^o = s_{21}^o$ and hence $|s_{12}^o| = |s_{21}^o|$, $\theta_{12} = \theta_{21}$ and

$$\beta = \theta_{11} + \theta_{22} - 2\theta_{12}. \quad (43)$$

Clearly, (41) implies that any four realizable magnitudes $|s_{11}^o|$, $|s_{21}^o|$, $|s_{12}^o|$ and $|s_{22}^o|$ satisfy the inequality

$$(1 - |s_{12}^o|^2)(1 - |s_{21}^o|^2) + (1 - |s_{11}^o|^2)(1 - |s_{22}^o|^2) \geq 1 - 2|s_{11}^o s_{22}^o s_{12}^o s_{21}^o|. \quad (44)$$

This corresponds to a choice of phases, for which $\beta = (2k+1)\pi$, k being an integer:

$$\theta_{11} + \theta_{22} - \theta_{12} - \theta_{21} = (2k + 1)\pi. \quad (45)$$

According to (41), the principal value of β cannot be *too* small. Its lower bound is determined from the equation

$$\cos\beta_{\min} = \frac{(1 - |s_{12}^o|^2)(1 - |s_{21}^o|^2) + (1 - |s_{11}^o|^2)(1 - |s_{22}^o|^2) - 1}{2|s_{11}^o s_{22}^o s_{12}^o s_{21}^o|}. \quad (46)$$

For lossless twoports, (41) simplifies considerably. Using (38) we find

$$\theta_{11} + \theta_{22} - \theta_{12} - \theta_{21} = (2k + 1)\pi. \quad (47)$$

It is easily verified that, in this case, the equality sign is attained in (44).

Let us return to the problem of determining λ_M and λ_m. Before embarking on the general analysis, it is advisable to dispose of a simple but important case first.

Case 1

At least one port matched. For this case, either Γ_g or Γ_l or both are zero. Suppose, to be definite, that $\Gamma_g = 0$. From (23), (27) and (28)

$$\lambda_M = 1 + |\Gamma_l s_{22}^o| \quad (48)$$

$$\lambda_m = 1 - |\Gamma_l s_{22}^o|. \quad (49)$$

Since

$$\lambda_m \lambda_M = 1 - |\Gamma_l s_{22}^o|^2 \leq 1,$$

$$|\epsilon|_{\max} = -20 \log(1 - |\Gamma_l s_{22}^o|). \quad (50)$$

To achieve (48), Γ_l must be 180° out of phase with s_{22}^o. In order to attain (49), Γ_l must be in phase with s_{22}^o. Both are compatible with physical realizability. Replacing log a by ln a in (50) we obtain for small $|\Gamma_l s_{22}^o|$

$$|\epsilon|_{\max} \approx 46|\Gamma_l s_{22}^o|. \quad (51)$$

The per cent decibel error is

$$\frac{|\epsilon|_{\max}}{A_o} \times 100 = -\frac{20 \log(1 - |\Gamma_l s_{22}^o|)}{A_o} \times 100. \quad (52)$$

It is not possible to express (52) solely in terms of the attenuator setting A_o unless $|s_{22}^o|$ is known as a function of A_o. Naturally, if $\Gamma_l = 0$ we simply replace Γ_l by Γ_g and s_{22}^o by s_{11}^o in (41)–(52).

Filters and *cutoff* attenuators are, to an excellent approximation, examples of dissipationless attenuators and are important not only at low frequencies but also in the microwave region. For them, a complete answer to the question of mismatched terminations is available.

Case 2

The attenuator N is *dissipationless*. The pertinent equations are (36), (37), (39) and (47). Making use of them we get

$$\Delta_o = |s_{11}^o s_{22}^o| e^{j(\theta_{11}+\theta_{22})} - |s_{12}^o s_{21}^o| e^{j(\theta_{12}+\theta_{21})} = e^{j(\theta_{11}+\theta_{22})}. \quad (53)$$

Set

$$\Gamma_g = |\Gamma_g| e^{j\beta_g}$$

and

$$\Gamma_l = |\Gamma_l| e^{j\beta_l}.$$

Then

$$\frac{D_o}{1-\Gamma_g\Gamma_l} = \frac{1-\eta[|\Gamma_g| e^{j(\beta_g+\theta_{11})} + |\Gamma_l| e^{j(\beta_l+\theta_{22})}] + |\Gamma_g\Gamma_l| e^{j(\theta_{11}+\theta_{22}+\beta_g+\beta_l)}}{1-|\Gamma_g\Gamma_l| e^{j(\beta_g+\beta_l)}} \tag{54}$$

where

$$\eta = |s_{11}^o| = |s_{22}^o| = (1-10^{-A_o/10})^{1/2}. \tag{55}$$

Of course, A_o represents the attenuator setting *in decibels*. In the form (54), realizability has been taken care of automatically and θ_{11}, θ_{22}, β_g, β_l are *arbitrary* angles. If we set

$$x = |\Gamma_g| e^{j(\beta_g+\theta_{11})}$$

and

$$y = |\Gamma_l| e^{j(\beta_l+\theta_{22})},$$

$$\frac{D_o}{1-\Gamma_g\Gamma_l} = \frac{1-\eta(x+y)+xy}{1-|\Gamma_g\Gamma_l| e^{j(\beta_g+\beta_l)}} \tag{56}$$

where x and y are complex numbers with arbitrary phases but whose magnitudes are $|\Gamma_g|$ and $|\Gamma_l|$, respectively. A moment's thought reveals that

$$\lambda_M = \frac{1+\eta(|\Gamma_g|+|\Gamma_l|)+|\Gamma_g\Gamma_l|}{1-|\Gamma_g\Gamma_l|} \tag{57}$$

and

$$\lambda_m = \frac{|1-\eta(x+y)+xy|_{\min}}{1+|\Gamma_g\Gamma_l|}. \tag{58}$$

Let us focus our attention on the quantity

$$u = 1-\eta(x+y)+xy = (x-\eta)y+1-\eta x. \tag{59}$$

For a fixed x, u is a linear transform of y. Consider the mapping

$$u = az+b. \tag{60}$$

Then

$$z = \frac{u-b}{a}$$

and the circle $|z|^2 = r^2$ goes into the circle

$$|ar|^2 = |u-b|^2 \tag{61}$$

whose center in the complex plane is b and whose radius is $|a|r$. From (59), the circle $|y|^2 = |\Gamma_l|^2$ is transformed into a circle with center located at $1-\eta x$ and radius $= |x-\eta|\cdot|\Gamma_g|$ (see Fig. 3).

We note that

$$|u|_{\max} = \overline{OC}+\overline{CB} = \overline{OC}+r = |1-\eta x|+r \tag{62}$$

$$|u|_{\min} = \overline{OC}-\overline{AC} = \overline{OC}-r = |1-\eta x|-r. \tag{63}$$

Hence,

$$|u|_{\max} = |1-\eta x| + |\Gamma_l|\cdot|x-\eta| \tag{64}$$

$$|u|_{\min} = |1-\eta x| - |\Gamma_l|\cdot|x-\eta|. \tag{65}$$

Now the magnitude of x is fixed at the value $|\Gamma_g|$ but its phase is *arbitrary*. Obviously, as we vary over the phase of x,

$$|u|_{\max,\max} = 1+\eta|\Gamma_g| + |\Gamma_l|(|\Gamma_g|+\eta)$$
$$= 1+\eta(|\Gamma_g|+|\Gamma_l|)+|\Gamma_g\Gamma_l|$$

as predicted earlier. The more difficult quantity to evaluate is $|u|_{\min,\min}$, the variation again being taken over the phase of x.

Let $x = |\Gamma_g| e^{j\theta}$. Then, substituting in (65) we get

$$|u|_{\min} = (1-2\eta|\Gamma_g|\cos\theta+\eta^2|\Gamma_g|^2)^{1/2} - |\Gamma_l|(|\Gamma_g|^2-2\eta|\Gamma_g|\cos\theta+\eta^2)^{1/2} \tag{66}$$

and the whole problem is reduced to finding the relative minima of (65) as a function of θ. This is a standard problem in the differential calculus which presents no particular difficulty. The details are relegated to Appendix III. Keeping in mind that

$$\lambda_m = \frac{|u|_{\min,\min}}{1+|\Gamma_g\Gamma_l|} \tag{66a}$$

the results may be summarized as follows:

1) If

$$\frac{||\Gamma_g|-|\Gamma_l||}{1-|\Gamma_g\Gamma_l|} \le \eta \le \frac{|\Gamma_g|+|\Gamma_l|}{1+|\Gamma_g\Gamma_l|} \tag{67}$$

$$\lambda_m^{-1} = \frac{1+|\Gamma_g\Gamma_l|}{\sqrt{(1-\eta^2)(1-|\Gamma_g|^2)(1-|\Gamma_l|^2)}}. \tag{68}$$

2) If

$$\eta \le \frac{||\Gamma_g|-|\Gamma_l||}{1-|\Gamma_g\Gamma_l|} \tag{69}$$

$$\lambda_m^{-1} = \frac{1+|\Gamma_g\Gamma_l|}{1-\eta||\Gamma_g|-|\Gamma_l||-|\Gamma_g\Gamma_l|}. \tag{70}$$

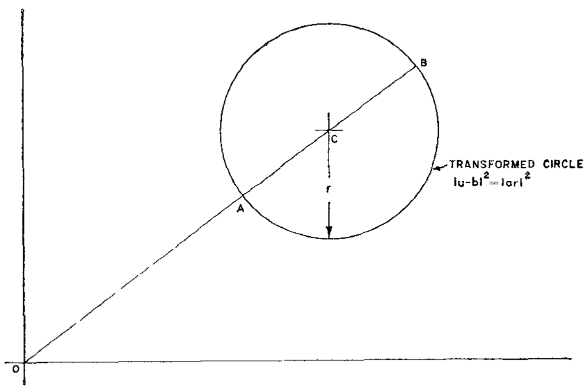


Fig. 3—Image of the circle $|y|^2 = |\Gamma_l|^2$.

3) If

$$\eta \geq \frac{|\Gamma_g| + |\Gamma_l|}{1 + |\Gamma_g\Gamma_l|} \quad (71)$$

$$\lambda_m^{-1} = \frac{1 + |\Gamma_g\Gamma_l|}{1 - \eta(|\Gamma_g| + |\Gamma_l|) + |\Gamma_g\Gamma_l|}. \quad (72)$$

In all cases,

$$|\epsilon|_{\max} = \text{largest } (20 \log \lambda_M, \ 20 \log \lambda_m^{-1}), \quad (73)$$

where

$$\lambda_M = \frac{1 + \eta(|\Gamma_g| + |\Gamma_l|) + |\Gamma_g\Gamma_l|}{1 - |\Gamma_g\Gamma_l|}. \quad (74)$$

Practical Applications

To facilitate the application of the above equations, graphs have been prepared and these are displayed in Figs. 4 to 10. Fig. 4 is a plot of η as a function of (A_o), the attenuator setting. Values of

$$\frac{|\Gamma_g| + |\Gamma_l|}{1 + |\Gamma_g\Gamma_l|}$$

and

$$\frac{||\Gamma_g| - |\Gamma_l||}{1 - |\Gamma_g\Gamma_l|}$$

are obtained from Figs. 5 and 6 which are plotted as a function of $|\Gamma_g|$. In both cases, $|\Gamma_l|$ is the parameter. Eqs. (68), (70), (72) and (74) are displayed in nomograph form in Figs. 7–10 respectively. Each chart contains a key. The numerical sequence indicates the order of steps. Fig. 8 is the only chart with a restriction. In using the extreme right hand side of the chart, if $|\Gamma_l| > |\Gamma_g|$, then the $|\Gamma_l|$ scale becomes $|\Gamma_g|$ and $|\Gamma_g|$ scale $|\Gamma_l|$.

To illustrate the use of these curves, consider the following example. A cutoff attenuator reads 10 db when operating between mismatched terminations specified by $|\Gamma_g| = 0.1$ and $|\Gamma_l| = 0.2$. Find the largest possible error consistent with physical realizability. The procedure is as follows. Figs. 4–6 provide the necessary information to determine which curve should be used to obtain the largest minimum error. For this example, Fig. 4 shows for $A_o = 10$ db, $\eta = 0.949$. For the values of $|\Gamma_g|$ and $|\Gamma_l|$ quoted, Fig. 5 yields

$$\frac{|\Gamma_g| + |\Gamma_l|}{1 + |\Gamma_g\Gamma_e|} = 0.294,$$

and Fig. 6 yields a value of 0.102 for

$$\frac{||\Gamma_g| - |\Gamma_l||}{1 - |\Gamma_g\Gamma_l|}.$$

Since $0.949 > 0.294$, we have Case 3 and (72) applies; therefore, Fig. 9 must be used. Step 1 is to connect $|\Gamma_l|$ to $|\Gamma_g|$ for a reference line. Draw a line through $\eta = 0.949$ parallel to the reference line and read $I_1 = 0.29$. Connect $|\Gamma_g|$ to $|\Gamma_l|$ in left side of chart and read $Q = 1.025$. Connect Q on upper scale to I_1 and read $\lambda_m^{-1} = 1.375$. To determine λ_M Fig. 10 is used. The first step is to connect $|\Gamma_g|$ and $|\Gamma_l|$ to obtain a reference line. Since this part of the chart is similar to Fig. 9, the same procedure is followed and a value of $I_1 = 0.29$ is obtained. Step 3 yields a value of 1.025 for Q, and step 4 a value of 0.975 for T. Connecting $Q' = Q$ on upper chart to I_2 yields a reference line. Drawing a line through T parallel to this reference line yields a value of 1.33 for λ_M. Hence, $\lambda_m^{-1} > \lambda_M$ and

$$|\epsilon|_{\max} = 20 \log (1.375) = 2.77 \text{ db},$$

which is quite large. As far as the actual attenuation A is concerned, it always lies between limits determined by A_o, λ_m and λ_M. From (13)

$$A_o - 20 \log \lambda_m^{-1} \leq A \leq A_o + 20 \log \lambda_M. \quad (75)$$

Using previous techniques, the problem proceeds as follows.

For the values quoted, the VSWR of the load and generator when they are not connected to the attenuator are 1.5 and 1.22, respectively. The maximum and minimum value of the VSWR of the input line when the output is terminated in a load $r_L = 1$ is given by

$$r_{\max} = r' r_L$$

and

$$r_{\min} = \begin{cases} \dfrac{r'}{r_L} & \text{for } r' > r_L \\ \dfrac{r_L}{r'} & \text{for } r_L > r' \end{cases}$$

where r' is the VSWR produced in the input line when the output is properly terminated; that is, $r_L = 1$. For a lossless 10 db attenuator $r' = 38.22$. Hence, $r_{\max} = 57.33$ and $r_{\min} = 25.48$. The error limits are evaluated from (8) of Beatty's paper[3] which is

$$\epsilon_{\min} = 20 \log \frac{(1 \pm |\Gamma_1\Gamma_g|)(1 \pm |S_{22}\Gamma_L|)}{1 \mp |\Gamma_g\Gamma_L|}.$$

In the above Γ_1 and S_{22} are the input reflection coefficients of the attenuator terminated in a load having a reflection coefficient of $\Gamma_L = 0.2$ and 0, respectively. The reflection coefficients corresponding to $r_{\max}$, $r_{\min}$, r', r_L and r_g are 0.966, 0.924, 0.949, 0.2 and 0.1, respectively. The maximum and minimum error limits using the above relation are 2.86 and 2.48 db, respectively. There is a slight difference in the larger value and, in general, a difference will always exist, *i.e.*, larger error limits arise from the use of previous existing techniques. It is evident that the use of the curves expedites the evaluation of error limits. In addition, it is usually not obvious which maximum will be the larger, depending upon the choice of signs used in (8) above, whereas this information can usually be obtained from the curves in this paper.

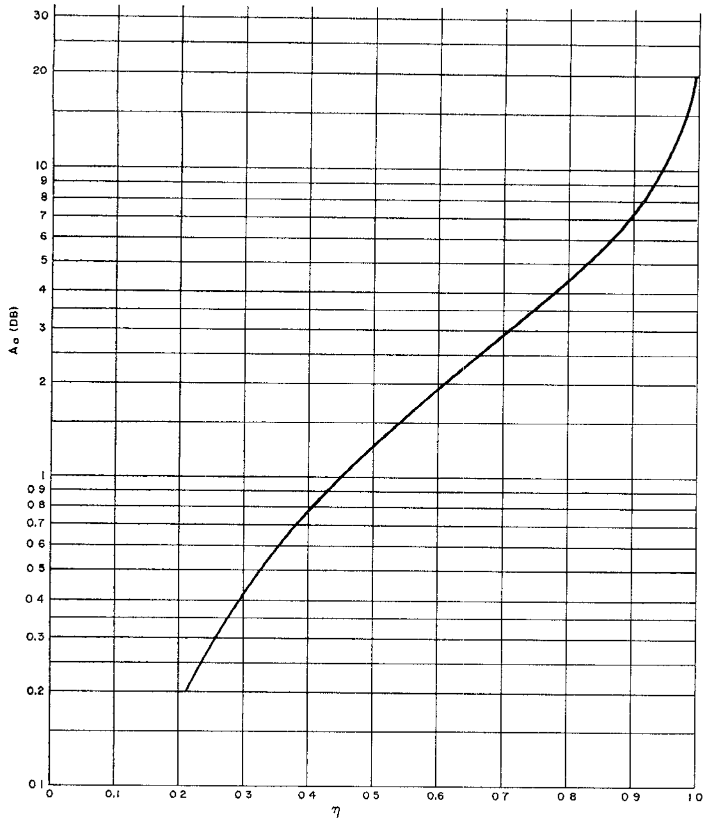
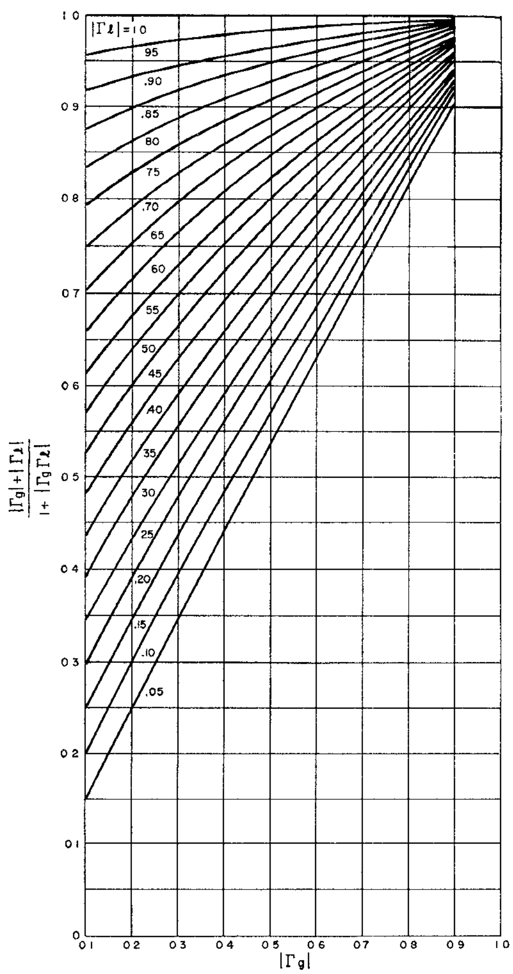
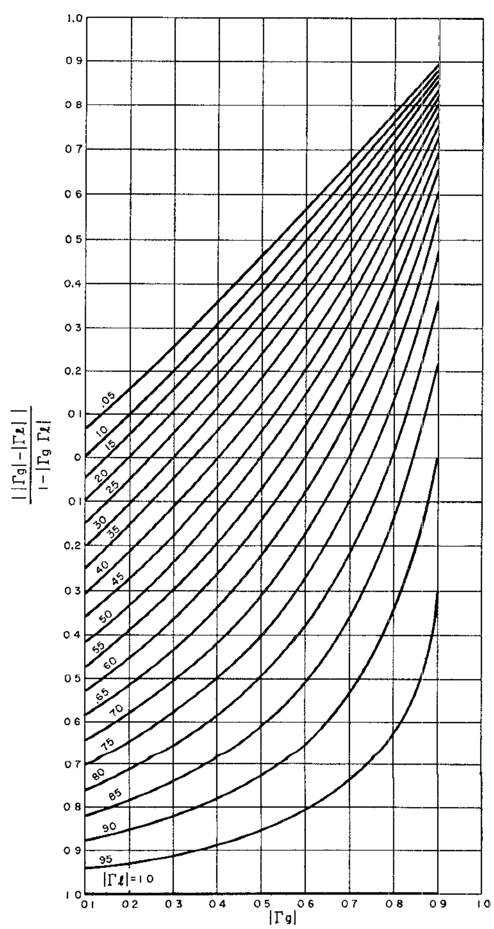


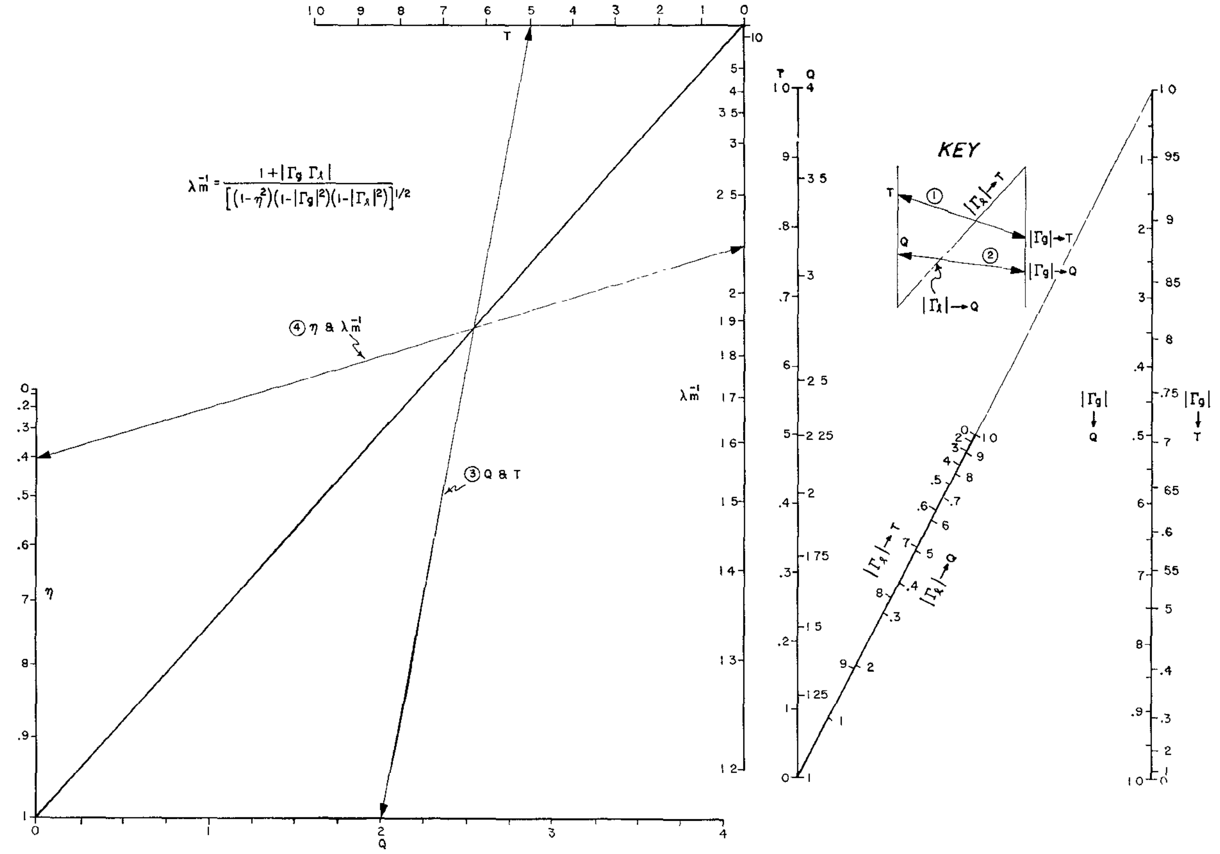
Fig. 4 (*left*)—Attenuator setting (A_o) vs η.

Fig. 5 (*below left*)—Values of
$$\frac{|\Gamma_g| + |\Gamma_l|}{1 + |\Gamma_g\Gamma_l|}$$
vs $|\Gamma_g|$, $|\Gamma_l|$ the parameter.

Fig. 6 (*below right*)—Values of
$$\frac{||\Gamma_g| - |\Gamma_l||}{1 - |\Gamma_g\Gamma_l|}$$
vs $|\Gamma_g|$, $|\Gamma_l|$ the parameter.

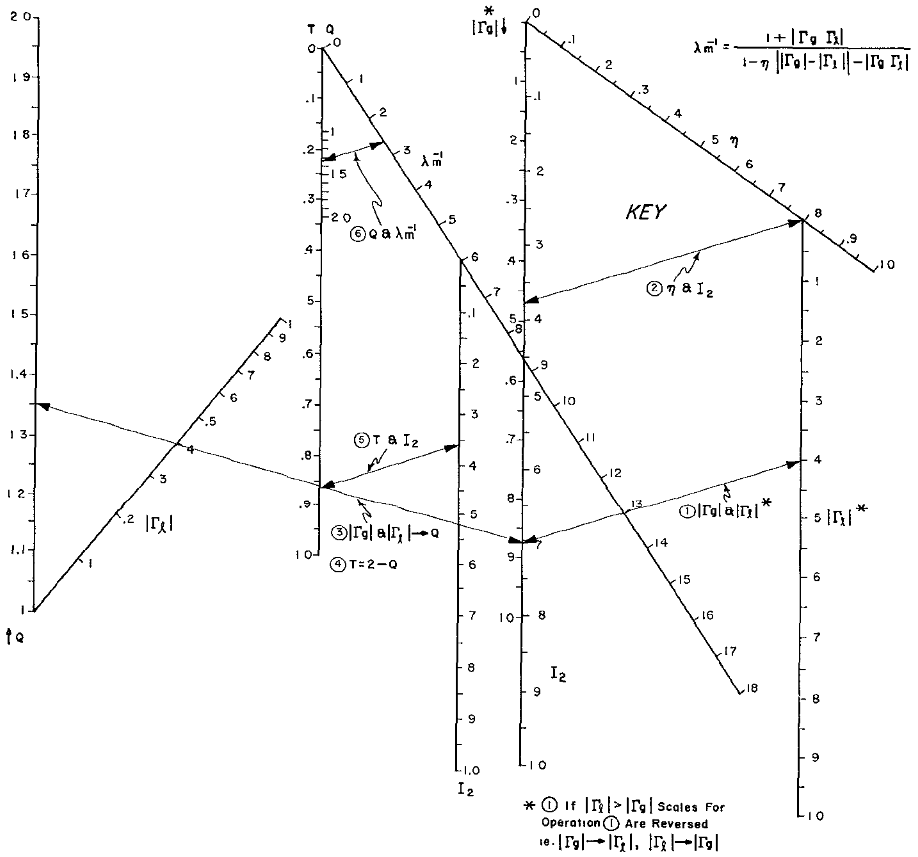






$\lambda_m^{-1} = \frac{1+|\Gamma_g \Gamma_\ell|}{[(1-\eta^2)(1-|\Gamma_g|^2)(1-|\Gamma_\ell|^2)]^{1/2}}$
KEY
① $|\Gamma_\ell| \to T$
$|\Gamma_g| \to T$
② $|\Gamma_g| \to Q$
$|\Gamma_\ell| \to Q$
③ Q & T
④ η & λ_m^{-1}

Fig. 7.



$\lambda_m^{-1} = \frac{1+|\Gamma_g \Gamma_\ell|}{1-\eta\,\big||\Gamma_g|-|\Gamma_\ell|\big|-|\Gamma_g \Gamma_\ell|}$
KEY
① $|\Gamma_g|$ & $|\Gamma_\ell|$ *
② η & I_2
③ $|\Gamma_g|$ & $|\Gamma_\ell| \to Q$
④ T=2−Q
⑤ T & I_2
⑥ Q & λ_m^{-1}
* ① If $|\Gamma_\ell| > |\Gamma_g|$ Scales For Operation ① Are Reversed
ie. $|\Gamma_g| \to |\Gamma_\ell|$, $|\Gamma_\ell| \to |\Gamma_g|$

Fig. 8.

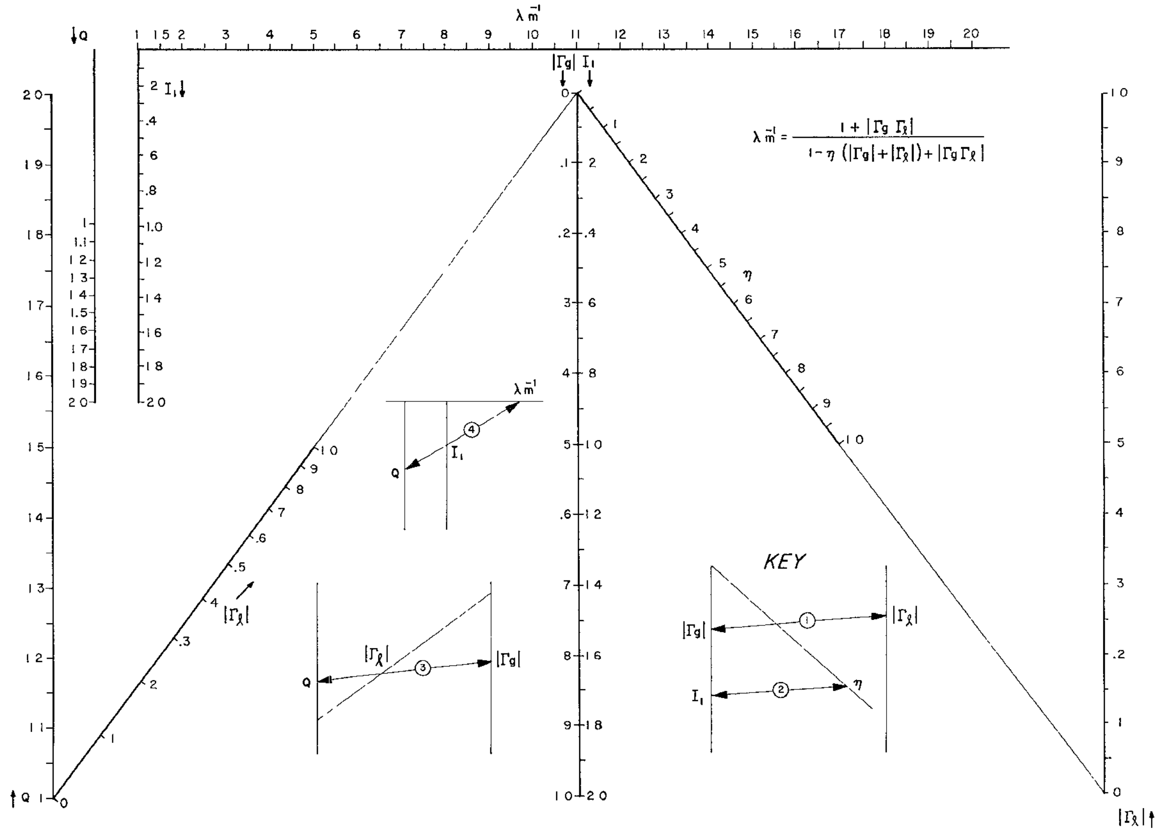

λ m⁻¹ = (1 + |Γg Γℓ|) / (1 − η (|Γg| + |Γℓ|) + |Γg Γℓ|)
KEY
|Γg|
|Γℓ|
I₁
η
Q

Fig. 9.

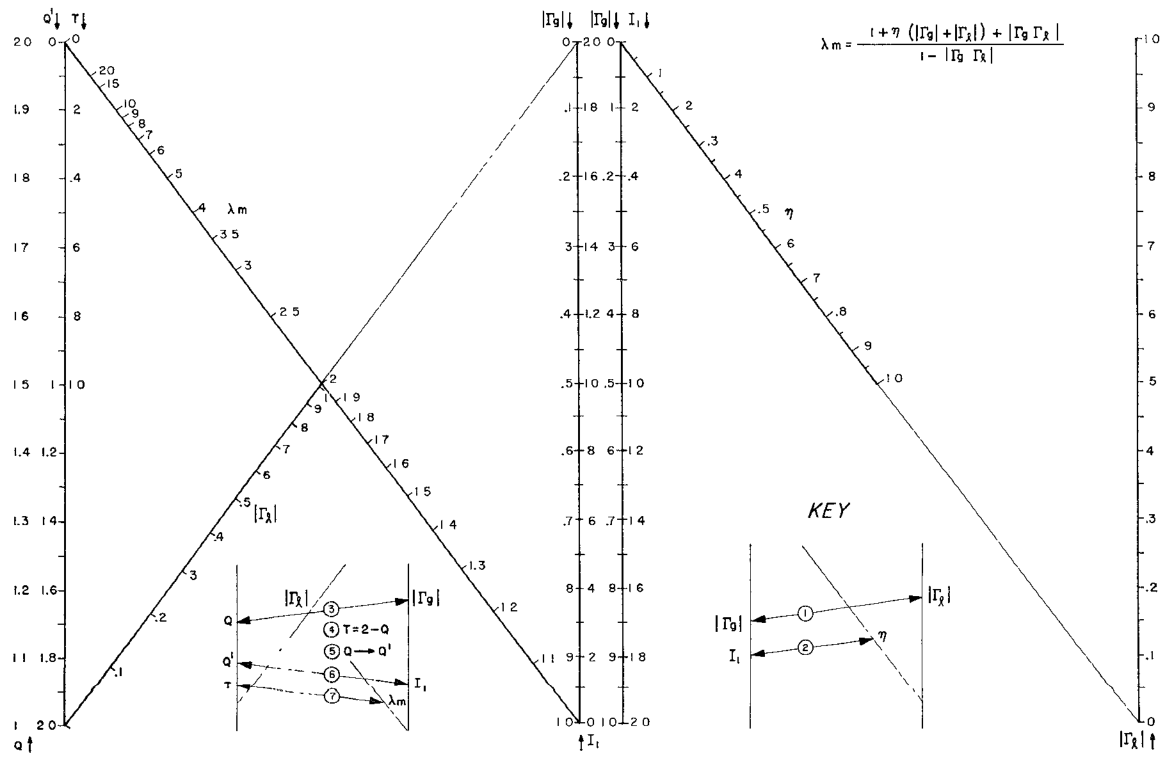

λ m = (1 + η (|Γg| + |Γℓ|) + |Γg Γℓ|) / (1 − |Γg Γℓ|)
KEY
T=2−Q
Q → Q'
λm
|Γℓ|
|Γg|
I₁
η

Fig. 10.

Appendix I

We begin by establishing (17). From (12) and (13) we obtain

$$z_g = \frac{2r_o}{1 - \Gamma_g} - \bar{z}_o \tag{76}$$

and

$$z_l = \frac{2r_o}{1 - \Gamma_l} - z_o \tag{77}$$

where

$$z_o \equiv r_o + jx_o.$$

From (76) and (77),

$$z_g + \bar{z}_g = 2r_o \frac{1 - |\Gamma_g|^2}{|1 - \Gamma_g|^2} \tag{78}$$

$$z_l + \bar{z}_l = 2r_o \frac{1 - |\Gamma_l|^2}{|1 - \Gamma_l|^2} \cdot \tag{79}$$

Again,

$$z_g + z_l = 2r_o \frac{1 - \Gamma_g \Gamma_l}{(1 - \Gamma_g)(1 - \Gamma_l)} \cdot \tag{80}$$

Substituting (78), (79) and (80) into the equation directly above (17) immediately yields (17), Q.E.D.

Appendix II

Before showing this relationship, we first introduce matrix notation which is consistent with Beatty.[3] If A is an arbitrary matrix, then A', $\overline{A'} = A^*$, det A and A^{-1} denote the transpose, the complex conjugate transpose (also called the adjoint), the determinant and the inverse of A, respectively. Column vectors are written $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}$, etc.

In the text since two sets of impedances, z_{01}, z_{02} and z_z and z_L were used two different scattering matrices S_o and S were required to describe N. To determine a relationship between S and S_o we begin with expressions (1) and (2). By definition

$$\boldsymbol{b}_o = S_o \boldsymbol{a}_o \tag{81}$$

$$\boldsymbol{b} = S\boldsymbol{a}. \tag{82}$$

From (1) and (2) the following equations (in matrix form) are obtained:

$$2R_o^{1/2}\boldsymbol{a}_o = \boldsymbol{V} + Z_o \boldsymbol{I} \tag{83}$$

$$2R_o^{1/2}\boldsymbol{b}_o = \boldsymbol{V} - Z_o^* \boldsymbol{I} \tag{84}$$

$$2R^{1/2}\boldsymbol{a} = \boldsymbol{V} + Z\boldsymbol{I} \tag{85}$$

$$2R^{1/2}\boldsymbol{b} = \boldsymbol{V} - Z^*\boldsymbol{I}. \tag{86}$$

Using (83) and the first formula below (16) in Youla[5] and (18) of the same paper results in

$$\begin{aligned} V + ZI &= \boldsymbol{V} + Z_o \boldsymbol{I} + (Z - Z_o)\boldsymbol{I} \\ &= 2R_o^{1/2}\boldsymbol{a}_o + (Z - Z_o) Y_{Ao} \boldsymbol{E}_o \\ &= [2R_o^{1/2} + (Z - Z_o)R_o^{-1/2}(1_n - S_o)]\boldsymbol{a}_o. \end{aligned} \tag{87}$$

Similarly,

$$\boldsymbol{V} - Z^*\boldsymbol{I} = [2R_o^{1/2} - (Z^* + Z_o)R_o^{-1/2}(1_n - S_o)]\boldsymbol{a}_o \tag{88}$$

where Y_{AO} and Y_A are the augmented admittance matrices corresponding to Z_o and Z, the impedance matrices which normalize S_o and S, respectively. From the above there results

$$2R_o^{1/2} + (Z - Z_o)R_o^{-1/2} = (Z + Z_o^*)R_o^{-1/2} \tag{89}$$

and

$$2R_o^{1/2} - (Z^* + Z_o)R_o^{1/2} = (Z_o^* - Z^*)R_o^{-1/2}. \tag{90}$$

Set

$$\Gamma = R_o^{1/2}(Z + Z_o^*)^{-1}(Z - Z_o)R_o^{-1/2} \tag{91}$$

or

$$= 1_n - 2R_o^{1/2}(Z + Z_o^*)^{-1}R_o^{1/2} \tag{92}$$

$$\Gamma = R_o^{-1/2}(Z - Z_o)(Z + Z_o^*)^{-1}R_o^{1/2}. \tag{93}$$

From (82)

$$\boldsymbol{V} - Z^*\boldsymbol{I} = R^{1/2}SR^{1/2}(\boldsymbol{V} + Z\boldsymbol{I}). \tag{94}$$

Using (91) and (94)

$$\begin{aligned} &(Z^* + Z_o)R_o^{-1/2}(S_o - \Gamma^*) \\ &= R^{1/2}SR^{-1/2}(Z + Z_o^*)R_o^{-1/2}(1_n - \Gamma S_o). \end{aligned} \tag{95}$$

From (92) and (93)

$$(Z + Z_o^*)R_o^{-1/2} = 2R_o^{1/2}(1_n - \Gamma)^{-1} \tag{96}$$

and

$$(Z^* + Z_o)R_o^{-1/2} = 2R_o^{1/2}(1_n - \Gamma^*)^{-1}. \tag{97}$$

Finally,

$$\begin{aligned} S = {} & R^{-1/2}R_o^{1/2}(1_n - \Gamma^*)^{-1}(S_o - \Gamma^*)(1_n - \Gamma S_o)^{-1} \\ & \cdot (1_n - \Gamma)R_o^{-1/2}R^{1/2}, \end{aligned} \tag{98}$$

the desired transformation. For a two-port

$$R_o = \text{diag}\,[r_{o_1}, r_{o_2}]$$

$$Z_o = \text{diag}\,[Z_{o_1}, Z_{o_2}]$$

$$\Gamma = \text{diag}\,[\Gamma_g, \Gamma_l]$$

and expressions 18 to 21 in the text follow.

Appendix III

Our next task is to derive (67)–(72). To find the minima of $|u|_{\min}$ it is necessary to set the first θ derivative of (66) equal to zero. Thus,

$$\frac{\sin\theta}{\sqrt{1-2\eta|\Gamma_g|\cos\theta+\eta^2|\Gamma_g|^2}} = \frac{|\Gamma_l|\sin\theta}{\sqrt{|\Gamma_g|^2-2\eta|\Gamma_g|\cos\theta+\eta^2}}. \quad (99)$$

The solutions are

$$\theta = 0, \quad \pi$$

and

$$\cos\theta = \frac{\eta^2(1-|\Gamma_g\Gamma_l|^2)+|\Gamma_g|^2-|\Gamma_l|^2}{2\eta|\Gamma_g|(1-|\Gamma_l|^2)}. \quad (100)$$

The corresponding $|u|_{\min}$ are

$\theta = 0$:

$$|u|_{\min} = 1-\eta|\Gamma_g| - |\Gamma_l|\cdot\big|\eta-|\Gamma_g|\big|. \quad (101)$$

$\theta = \pi$:

$$|u|_{\min} = 1+\eta(|\Gamma_g|-|\Gamma_l|)-|\Gamma_g\Gamma_l| \quad (102)$$

(100):

$$|u|_{\min} = \sqrt{(1-\eta^2)(1-|\Gamma_g|^2)(1-|\Gamma_l|^2)}. \quad (103)$$

To decide when these values are actually minima, we must calculate the second θ derivative of $|u|_{\min}$ and evaluate it at the respective points. Omitting the details we have the following results:

$\theta = 0$:

$$\frac{d^2|u|_{\min}}{d\theta^2} = \eta|\Gamma_g|\left(\frac{1}{1-\eta|\Gamma_g|}-\frac{|\Gamma_l|}{\big|\eta-|\Gamma_g|\big|}\right) \quad (104)$$

$\theta = \pi$:

$$\frac{d^2|u|_{\min}}{d\theta^2} = -\eta|\Gamma_g|\left(\frac{1}{1+\eta|\Gamma_g|}-\frac{|\Gamma_l|}{\eta+|\Gamma_g|}\right) \quad (105)$$

(100):

$$\frac{d^2|u|_{\min}}{d\theta^2} = k|\Gamma_l|(1-|\Gamma_l|^2) \geq 0, \quad (106)$$

k being a positive constant.

Consequently, (100) always leads to a minimum. Eq. (104) reveals that $\theta=0$ yields a minimum whenever

$$|\Gamma_l| \leq \frac{\big|\eta-|\Gamma_g|\big|}{1-\eta|\Gamma_g|} \quad (107)$$

and a maximum otherwise. Similarly $\theta=\pi$ yields a *minimum* when

$$|\Gamma_l| > \frac{\eta+|\Gamma_g|}{1+\eta|\Gamma_g|} \quad (108)$$

and a maximum otherwise. The value of θ given by (100) *always* results in a *minimum*. To summarize,

1) If

$$\frac{\big|\eta-|\Gamma_g|\big|}{1-\eta|\Gamma_g|} \leq |\Gamma_l| \leq \frac{\eta+|\Gamma_g|}{1+\eta|\Gamma_g|}, \quad (109)$$

$$\lambda_m^{-1} = \frac{1+|\Gamma_g\Gamma_l|}{\sqrt{(1-\eta^2)(1-|\Gamma_g|^2)(1-|\Gamma_l|^2)}}. \quad (110)$$

2) If

$$|\Gamma_l| \leq \frac{\big|\eta-|\Gamma_g|\big|}{1-\eta|\Gamma_g|}, \quad (111)$$

$$\lambda_m^{-1} = \frac{1+|\Gamma_g\Gamma_l|}{1-\eta|\Gamma_g|-|\Gamma_l|\cdot\big|\eta-|\Gamma_g|\big|}. \quad (112)$$

3) If

$$|\Gamma_l| \geq \frac{\eta+|\Gamma_g|}{1+\eta|\Gamma_g|}. \quad (113)$$

$$\lambda_m^{-1} = \frac{1+|\Gamma_g\Gamma_l|}{1+\eta(|\Gamma_g|-|\Gamma_l|)-|\Gamma_g\Gamma_l|}. \quad (114)$$

As they stand, (109)–(114) are unsymmetric in $|\Gamma_g|$ and $|\Gamma_l|$. To remedy this defect, we argue as follows. Let $|\Gamma_g|$, $|\Gamma_l|$ and η be three values consistent with (91). Consider the bilinear transformation

$$w = \frac{\xi-|\Gamma_g|}{1-\xi|\Gamma_g|}. \quad (115)$$

As ξ describes the circle $|\xi|=\eta$, w also describes a circle and furthermore

$$\frac{\big|\eta-|\Gamma_g|\big|}{1-\eta|\Gamma_g|} \leq |w| \leq \frac{\eta+|\Gamma_g|}{1+\eta|\Gamma_g|};$$

i.e., $|w|$ yields all the values $|\Gamma_g|$ compatible with (109). Let $|w|=|\Gamma_l|$ denote one of these. Solving (115) for ξ gives

$$\xi = \frac{w+|\Gamma_g|}{1+w|\Gamma_g|}. \quad (116)$$

As w describes the circle $|w|=|\Gamma_l|$, ξ also describes a circle and on this circle

$$\frac{\big||\Gamma_g|-|\Gamma_l|\big|}{1-|\Gamma_g\Gamma_l|} \leq |\xi| = \eta \leq \frac{|\Gamma_g|+|\Gamma_l|}{1+|\Gamma_g\Gamma_l|}. \quad (117)$$

That is to say, for a prescribed $|\Gamma_g|$ and $|\Gamma_l|$, (117) yields all η compatible with (109). In short, (109) and (117) are equivalent. Following this reasoning to its logical conclusion, we obtain (67)–(72), Q.E.D.